

F04FEF – NAG Fortran Library Routine Document

Note. Before using this routine, please read the Users' Note for your implementation to check the interpretation of bold italicised terms and other implementation-dependent details.

1 Purpose

F04FEF solves the Yule–Walker equations for a real symmetric positive-definite Toeplitz system.

2 Specification

```
SUBROUTINE F04FEF(N, T, X, WANTP, P, WANTV, V, VLAST, WORK, IFAIL)
  INTEGER          N, IFAIL
  real            T(0:N), X(*), P(*), V(*), VLAST, WORK(*)
  LOGICAL          WANTP, WANTV
```

3 Description

This routine solves the equations

$$Tx = -t,$$

where T is the n by n symmetric positive-definite Toeplitz matrix

$$T = \begin{pmatrix} \tau_0 & \tau_1 & \tau_2 & \cdots & \tau_{n-1} \\ \tau_1 & \tau_0 & \tau_1 & \cdots & \tau_{n-2} \\ \tau_2 & \tau_1 & \tau_0 & \cdots & \tau_{n-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \tau_{n-1} & \tau_{n-2} & \tau_{n-3} & \cdots & \tau_0 \end{pmatrix}$$

and t is the vector

$$t^T = (\tau_1 \tau_2 \cdots \tau_n).$$

The routine uses the method of Durbin [4],[5]. Optionally the mean square prediction errors and/or the partial correlation coefficients for each step can be returned.

4 References

- [1] Bunch J R (1985) Stability of methods for solving Toeplitz systems of equations *SIAM J. Sci. Statist. Comput.* **6** 349–364
- [2] Bunch J R (1987) The weak and strong stability of algorithms in numerical linear algebra *Linear Algebra Appl.* **88/89** 49–66
- [3] Cybenko G (1980) The numerical stability of the Levinson–Durbin algorithm for Toeplitz systems of equations *SIAM J. Sci. Statist. Comput.* **1** 303–319
- [4] Durbin J (1960) The fitting of time series models *Rev. Inst. Internat. Stat.* **28** 233
- [5] Golub G H and van Loan C F (1996) *Matrix Computations* Johns Hopkins University Press (3rd Edition), Baltimore

5 Parameters

- 1: N — INTEGER *Input*
On entry: the order of the Toeplitz matrix T .
Constraint: $N \geq 0$. When $N = 0$, then an immediate return is effected.

- 2:** T(0:N) — *real* array Input
On entry: T(0) must contain the value τ_0 of the diagonal elements of T , and the remaining N elements of T must contain the elements of the vector t .
Constraint: T(0) > 0.0. Note that if this is not true, then the Toeplitz matrix cannot be positive-definite.
- 3:** X(*) — *real* array Output
Note: the dimension of the array X must be at least max(1,N).
On exit: the solution vector x .
- 4:** WANTP — LOGICAL Input
On entry: WANTP must be set to .TRUE. if the partial (auto)correlation coefficients are required, and must be set .FALSE. otherwise.
- 5:** P(*) — *real* array Output
Note. If WANTP = .TRUE., the dimension of the array P must be at least max(1,N); otherwise the dimension must be at least 1.
On exit: with WANTP as .TRUE., the i th element of P contains the partial (auto)correlation coefficient, or reflection coefficient, p_i for the i th step. (See Section 8 and the G13 Chapter Introduction.) If WANTP is .FALSE., then P is not referenced. Note that in any case, $x_n = p_n$.
- 6:** WANTV — LOGICAL Input
On entry: WANTV must be set to .TRUE. if the mean square prediction errors are required, and must be set .FALSE. otherwise.
- 7:** V(*) — *real* array Output
Note. If WANTV = .TRUE., the dimension of the array V must be at least max(1,N); otherwise the dimension must be at least 1.
On exit: with WANTV as .TRUE., the i th element of V contains the mean square prediction error, or predictor error variance ratio, v_i , for the i th step. (See Section 8 and the G13 Chapter Introduction.) If WANTV is .FALSE., then V is not referenced.
- 8:** VLAST — *real* Output
On exit: the value of v_n , the mean square prediction error for the final step.
- 9:** WORK(*) — *real* array Workspace
Note: the dimension of the array $WORK$ must be at least max(1,N-1).
- 10:** IFAIL — INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.
On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL \neq 0 on exit, users are recommended to set IFAIL to -1 before entry. **It is then essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings specified by the routine:

IFAIL = -1

On entry, $N < 0$,
or $T(0) \leq 0.0$.

IFAIL > 0

The principal minor of order (IFAIL+1) of the Toeplitz matrix is not positive-definite to working accuracy. If, on exit, x_{IFAIL} is close to unity, then the principal minor was close to being singular, and the sequence $\tau_0, \tau_1, \dots, \tau_{\text{IFAIL}}$ may be a valid sequence nevertheless. The first IFAIL elements of X return the solution of the equations

$$T_{\text{IFAIL}}x = -(\tau_1, \tau_2, \dots, \tau_{\text{IFAIL}})^T,$$

where T_{IFAIL} is the IFAILth principal minor of T . Similarly, if WANTP and/or WANTV are true, then P and/or V return the first IFAIL elements of P and V respectively and VLAST returns v_{IFAIL} . In particular if IFAIL = N, then the solution of the equations $Tx = -t$ is returned in X, but τ_N is such that T_{N+1} would not be positive-definite to working accuracy.

7 Accuracy

The computed solution of the equations certainly satisfies

$$r = Tx + t,$$

where $\|r\|_1$ is approximately bounded by

$$\|r\|_1 \leq c\epsilon \left(\prod_{i=1}^n (1 + |p_i|) - 1 \right),$$

c being a modest function of n and ϵ being the *machine precision*. This bound is almost certainly pessimistic, but it has not yet been established whether or not the method of Durbin is backward stable. If $|p_n|$ is close to one, then the Toeplitz matrix is probably ill-conditioned and hence only just positive-definite. For further information on stability issues see Bunch [1] and [2], Cybenko [3] and Golub and Van Loan [5]. The following bounds on $\|T^{-1}\|_1$ hold,

$$\max \left(\frac{1}{v_{n-1}}, \frac{1}{\prod_{i=1}^{n-1} (1 - p_i)} \right) \leq \|T^{-1}\|_1 \leq \prod_{i=1}^{n-1} \left(\frac{1 + |p_i|}{1 - |p_i|} \right).$$

Note. $v_n < v_{n-1}$. The norm of T^{-1} may also be estimated using routine F04YCF.

8 Further Comments

The number of floating-point operations used by this routine is approximately $2n^2$, independent of the values of WANTP and WANTV.

The mean square prediction error, v_i , is defined as

$$v_i = (\tau_0 + (\tau_1\tau_2 \dots \tau_{i-1})y_{i-1})/\tau_0,$$

where y_i is the solution of the equations

$$T_i y_i = -(\tau_1\tau_2 \dots \tau_i)^T$$

and the partial correlation coefficient, p_i , is defined as the i th element of y_i . Note that $v_i = (1 - p_i^2)v_{i-1}$.

9 Example

To find the solution of the Yule–Walker equations $Tx = -t$, where

$$T = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

9.1 Program Text

Note. The listing of the example program presented below uses bold italicised terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```

*      F04FEF Example Program Text
*      Mark 15 Release. NAG Copyright 1991.
*      .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NMAX
      PARAMETER        (NMAX=100)
*      .. Local Scalars ..
      real            VLAST
      INTEGER          I, IFAIL, N
      LOGICAL          WANTP, WANTV
*      .. Local Arrays ..
      real            P(NMAX), T(0:NMAX), V(NMAX), WORK(NMAX-1),
+                   X(NMAX)
*      .. External Subroutines ..
      EXTERNAL         F04FEF
*      .. Executable Statements ..
      WRITE (NOUT,*) 'F04FEF Example Program Results'
*      Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N
      WRITE (NOUT,*)
      IF ((N.LT.0) .OR. (N.GT.NMAX)) THEN
         WRITE (NOUT,99999) 'N is out of range. N = ', N
      ELSE
         READ (NIN,*) (T(I),I=0,N)
         WANTP = .TRUE.
         WANTV = .TRUE.
*
         IFAIL = -1
*
         CALL F04FEF(N,T,X,WANTP,P,WANTV,V,VLAST,WORK,IFAIL)
*
         IF (IFAIL.EQ.0) THEN
            WRITE (NOUT,*)
            WRITE (NOUT,*) 'Solution vector'
            WRITE (NOUT,99998) (X(I),I=1,N)
            IF (WANTP) THEN
               WRITE (NOUT,*)
               WRITE (NOUT,*) 'Reflection coefficients'
               WRITE (NOUT,99998) (P(I),I=1,N)
            END IF
            IF (WANTV) THEN
               WRITE (NOUT,*)
               WRITE (NOUT,*) 'Mean square prediction errors'
            END IF
         END IF
      END IF

```

```

        WRITE (NOUT,99998) (V(I),I=1,N)
    END IF
ELSE IF (IFAIL.GT.0) THEN
    WRITE (NOUT,*)
    WRITE (NOUT,99999) 'Solution for system of order', IFAIL
    WRITE (NOUT,99998) (X(I),I=1,IFAIL)
    IF (WANTP) THEN
        WRITE (NOUT,*)
        WRITE (NOUT,*) 'Reflection coefficients'
        WRITE (NOUT,99998) (P(I),I=1,IFAIL)
    END IF
    IF (WANTV) THEN
        WRITE (NOUT,*)
        WRITE (NOUT,*) 'Mean square prediction errors'
        WRITE (NOUT,99998) (V(I),I=1,IFAIL)
    END IF
END IF
END IF
STOP
*
99999 FORMAT (1X,A,I5)
99998 FORMAT (1X,5F9.4)
END

```

9.2 Program Data

F04FEF Example Program Data

```

4                               :Value of N
4.0 3.0 2.0 1.0 0.0           :End of vector T

```

9.3 Program Results

F04FEF Example Program Results

```

Solution vector
-0.8000  0.0000  0.0000  0.2000

```

```

Reflection coefficients
-0.7500  0.1429  0.1667  0.2000

```

```

Mean square prediction errors
0.4375  0.4286  0.4167  0.4000

```
