S11AAF – NAG Fortran Library Routine Document

Note. Before using this routine, please read the Users' Note for your implementation to check the interpretation of bold italicised terms and other implementation-dependent details.

1 Purpose

S11AAF returns the value of the inverse hyperbolic tangent, $\operatorname{arctanh} x$, via the routine name.

2 Specification

```
real FUNCTION S11AAF(X, IFAIL)
INTEGER              IFAIL
real                 X
```

3 Description

The routine calculates an approximate value for the inverse hyperbolic tangent of its argument, $\operatorname{arctanh} x$.

For $x^2 \leq \frac{1}{2}$ it is based on the Chebyshev expansion

$$\operatorname{arctanh} x = x \times y(t) = x \sum_{r=0}^{\prime} a_r T_r(t)$$

where $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$, $\quad -1 \leq t \leq 1, \quad$ and $\; t = 4x^2 - 1$.

For $\frac{1}{2} < x^2 < 1$, it uses

$$\operatorname{arctanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

For $|x| \geq 1$, the routine fails as $\operatorname{arctanh} x$ is undefined.

4 References

[1] Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* Dover Publications (3rd Edition)

5 Parameters

1: X — ***real*** *Input*

On entry: the argument x of the function.

Constraint: $|\mathrm{X}| < 1.0$.

2: IFAIL — INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in Chapter P01) the recommended value is 0.

On exit: IFAIL $= 0$ unless the routine detects an error (see Section 6).

6 Error Indicators and Warnings

Errors detected by the routine:

IFAIL $= 1$

The routine has been called with an argument greater than or equal to 1.0 in magnitude, for which arctanh is not defined. The result is returned as zero.

7 Accuracy

If δ and ϵ are the relative errors in the argument and result, respectively, then in principle

$$|\epsilon| \simeq \left| \frac{x}{(1-x^2)\operatorname{arc}\tanh x} \times \delta \right|.$$

That is, the relative error in the argument, x, is amplified by at least a factor $\frac{x}{(1-x^2)\operatorname{arc}\tanh x}$ in the result. The equality should hold if δ is greater than the ***machine precision*** (δ due to data errors etc.) but if δ is simply due to round-off in the machine representation then it is possible that an extra figure may be lost in internal calculation round-off.

The behaviour of the amplification factor is shown in the following graph:

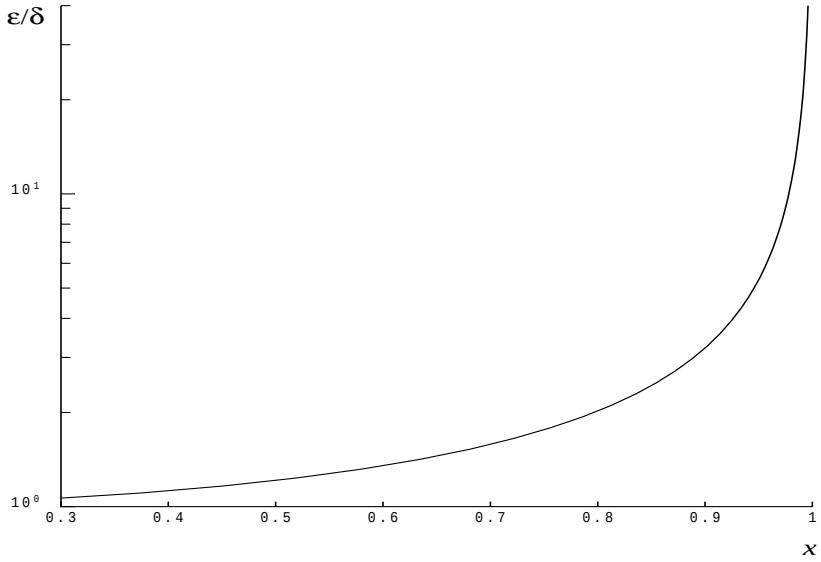

Figure 1

The factor is not significantly greater than one except for arguments close to $|x| = 1$. However in the region where $|x|$ is close to one, $1 - |x| \sim \delta$, the above analysis is inapplicable since x is bounded by definition, $|x| < 1$. In this region where arctanh is tending to infinity we have

$$\epsilon \sim 1/\ln \delta$$

which implies an obvious, unavoidable serious loss of accuracy near $|x| \sim 1$, e.g., if x and 1 agree to 6 significant figures, the result for $\operatorname{arctanh} x$ would be correct to at most about one figure.

8 Further Comments

None.

9 Example

The following program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

9.1 Program Text

Note. The listing of the example program presented below uses bold italicised terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     S11AAF Example Program Text
*     Mark 14 Revised.  NAG Copyright 1989.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
*     .. Local Scalars ..
      real             X, Y
      INTEGER          IFAIL
*     .. External Functions ..
      real             S11AAF
      EXTERNAL         S11AAF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'S11AAF Example Program Results'
*     Skip heading in data file
      READ (NIN,*)
      WRITE (NOUT,*)
      WRITE (NOUT,*) '     X           Y        IFAIL'
      WRITE (NOUT,*)
   20 READ (NIN,*,END=40) X
      IFAIL = 1
*
      Y = S11AAF(X,IFAIL)
*
      WRITE (NOUT,99999) X, Y, IFAIL
      GO TO 20
   40 STOP
*
99999 FORMAT (1X,1P,2e12.3,I7)
      END
```

9.2 Program Data

```
S11AAF Example Program Data
           -0.5
            0.0
            0.5
           -0.9999
            3.0
```

9.3 Program Results

```
 S11AAF Example Program Results

      X           Y        IFAIL

  -5.000E-01  -5.493E-01      0
   0.000E+00   0.000E+00      0
   5.000E-01   5.493E-01      0
  -9.999E-01  -4.952E+00      0
   3.000E+00   0.000E+00      1
```