S21BCF – NAG Fortran Library Routine Document

Note. Before using this routine, please read the Users' Note for your implementation to check the interpretation of bold italicised terms and other implementation-dependent details.

1 Purpose

S21BCF returns a value of the symmetrised elliptic integral of the second kind, via the routine name.

2 Specification

```
real FUNCTION S21BCF(X, Y, Z, IFAIL)
INTEGER             IFAIL
real                X, Y, Z
```

3 Description

This routine calculates an approximate value for the integral

$$R_D(x,y,z) = \frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}$$

where x, $y \geq 0$, at most one of x and y is zero, and $z > 0$.

The basic algorithm, which is due to Carlson [2] and [3], is to reduce the arguments recursively towards their mean by the rule:

$$\begin{aligned}
x_0 &= x, y_0 = y, z_0 = z \\
\mu_n &= (x_n + y_n + 3z_n)/5 \\
X_n &= (1 - x_n)/\mu_n \\
Y_n &= (1 - y_n)/\mu_n \\
Z_n &= (1 - z_n)/\mu_n \\
\lambda_n &= \sqrt{x_n y_n} + \sqrt{y_n z_n} + \sqrt{z_n x_n} \\
x_{n+1} &= (x_n + \lambda_n)/4 \\
y_{n+1} &= (y_n + \lambda_n)/4 \\
z_{n+1} &= (z_n + \lambda_n)/4
\end{aligned}$$

For n sufficiently large,

$$\epsilon_n = \max(|X_n|, |Y_n|, |Z_n|) \sim \left(\frac{1}{4}\right)^n$$

and the function may be approximated adequately by a 5th order power series

$$\begin{aligned}
R_D(x,y,z) = \; & 3\sum_{m=0}^{n-1} \frac{4^{-m}}{(z_m + \lambda_n)\sqrt{z_m}} \\
& + \frac{4^{-n}}{\sqrt{\mu_n^3}}\left[1 + \frac{3}{7}S_n^{(2)} + \frac{1}{3}S_n^{(3)} + \frac{3}{22}(S_n^{(2)})^2 + \frac{3}{11}S_n^{(4)} + \frac{3}{13}S_n^{(2)}S_n^{(3)} + \frac{3}{13}S_n^{(5)}\right]
\end{aligned}$$

where $S_n^{(m)} = (X_n^m + Y_n^m + 3Z_n^m)/2m$. The truncation error in this expansion is bounded by $\frac{3\epsilon_n^6}{\sqrt{(1-\epsilon_n)^3}}$ and the recursive process is terminated when this quantity is negligible compared with the ***machine precision***.

The routine may fail either because it has been called with arguments outside the domain of definition, or with arguments so extreme that there is an unavoidable danger of setting underflow or overflow.

Note. $R_D(x,x,x) = x^{-3/2}$, so there exists a region of extreme arguments for which the function value is not representable.

4 References

[1] Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* Dover Publications (3rd Edition)

[2] Carlson B C (1978) Computing elliptic integrals by duplication *Preprint* Department of Physics, Iowa State University

[3] Carlson B C (1988) A table of elliptic integrals of the third kind *Math. Comput.* **51** 267–280

5 Parameters

1: X — ***real*** *Input*

2: Y — ***real*** *Input*

3: Z — ***real*** *Input*

On entry: the arguments x, y and z of the function.

Constraint: X, Y ≥ 0.0, Z > 0.0 and only one of X and Y may be zero.

4: IFAIL — INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in Chapter P01) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors detected by the routine:

IFAIL = 1

On entry, either X or Y is negative, or both X and Y are zero; the function is undefined.

IFAIL = 2

On entry, Z ≤ 0.0; the function is undefined.

IFAIL = 3

On entry, either Z is too close to zero or both X and Y are too close to zero: there is a danger of setting overflow.

IFAIL = 4

On entry, at least one of X, Y and Z is too large: there is a danger of setting underflow. On soft failure the routine returns zero.

7 Accuracy

In principle the routine is capable of producing full ***machine precision***. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the ***machine precision***.

8 Further Comments

Users should consult the Chapter Introduction which shows the relationship of this function to the classical definitions of the elliptic integrals.

9 Example

This example program simply generates a small set of non-extreme arguments which are used with the routine to produce the table of low accuracy results.

9.1 Program Text

Note. The listing of the example program presented below uses bold italicised terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     S21BCF Example Program Text
*     Mark 14 Revised.  NAG Copyright 1989.
*     .. Parameters ..
      INTEGER          NOUT
      PARAMETER        (NOUT=6)
*     .. Local Scalars ..
      real             RD, X, Y, Z
      INTEGER          IFAIL, IX, IY
*     .. External Functions ..
      real             S21BCF
      EXTERNAL         S21BCF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'S21BCF Example Program Results'
      WRITE (NOUT,*)
      WRITE (NOUT,*) '    X      Y      Z        S21BCF  IFAIL'
      WRITE (NOUT,*)
      DO 40 IX = 1, 3
         X = IX*0.5e0
         DO 20 IY = IX, 3
            Y = IY*0.5e0
            Z = 1.0e0
            IFAIL = 1
*
            RD = S21BCF(X,Y,Z,IFAIL)
*
            WRITE (NOUT,99999) X, Y, Z, RD, IFAIL
   20    CONTINUE
   40 CONTINUE
      STOP
*
99999 FORMAT (1X,3F7.2,F12.4,I5)
      END
```

9.2 Program Data

None.

9.3 Program Results

```
 S21BCF Example Program Results

    X      Y      Z        S21BCF  IFAIL

   0.50   0.50   1.00      1.4787    0
   0.50   1.00   1.00      1.2108    0
   0.50   1.50   1.00      1.0611    0
   1.00   1.00   1.00      1.0000    0
   1.00   1.50   1.00      0.8805    0
   1.50   1.50   1.00      0.7775    0
```